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Asymptotic estimation of some multiple integrals and the electromagnetic deuteron form factors at high momentum transfer

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Abstract
A theorem about asymptotic estimation of multiple integrals of a special type is proved for the case when the integrand peaks at the integration domain boundary, but not at a point of extremum. Using this theorem, the asymptotic expansion of the electromagnetic deuteron form factors at high momentum transfers is obtained in the framework of a two-nucleon model in both the nonrelativistic and relativistic impulse approximations. It is found that the relativistic effects slow down the decrease of deuteron form factors and result in agreement between the relativistic asymptotics and experimental data at high momentum transfers.

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1. Introduction

Recent advances in experimental investigations of a hadron structure arouse the interest in the theoretical study of the hadron electromagnetic form factors at high momentum transfers (see, e.g., [1] and references therein). In this connection the JLab program of investigations on elastic electron–deuteron scattering experiments at $Q^2 \simeq 10$ (GeV c^{-1})^2 ($Q^2 = -q^2$, q is the transferred momentum) [2] attracts considerable attention. There exists a hope that these JLab experiments will help to determine the limits of application for the two-nucleon model and to clarify the interplay between nucleon–nucleon and quark approaches to the deuteron.

Using the asymptotic expansion presented in this paper we show [3] that the momentum transfer region in the JLab experiments is asymptotical for the deuteron considered as a nucleon–nucleon system. That is why the study of the electromagnetic deuteron form factors is interesting at $Q^2 \to \infty$.

The present work is devoted to the theoretical investigation of deuteron form factors asymptotic behavior at high momentum transfer in the framework of the two-nucleon model. The form factors asymptotics is studied in accordance with the following points.

(1) As a rule, the calculation of the form factors asymptotic behavior in the relativistic approaches reduces to the asymptotic estimation of n-tuple integrals. In the relativistic approach used in our work the deuteron form factors are expressed in terms of double integrals where integrands peak at the integration domain boundary, and the corresponding point is not a point of extremum. In this connection the theorem defining asymptotic expansion of n-tuple integrals with such integrands is proven in our paper.

(2) In general, high momentum transfers require relativistic consideration. However we begin the consideration of the asymptotic estimation of electromagnetic deuteron form factors with the nonrelativistic case and the nonrelativistic impulse approximation at $Q^2 \to \infty$. This is due to the facts that, first, the nonrelativistic calculation is less complicated and, second, this calculation is important for the establishment of the role of the relativistic effects.

(3) The asymptotic expansion of the relativistic deuteron form factors is calculated in the relativistic invariant impulse approximation in a variant of instant form of Poincaré-invariant quantum mechanics (PIQM) developed in our papers previously [4–8]. The relativistic calculations are performed by analogy with the nonrelativistic case mentioned in paragraph 2. It is shown that relativistic effects essentially slow down the asymptotical decrease of the form factors.

(4) It is found that relativistic asymptotics obtained in the framework of the two-nucleon model coincides with the experimental data.

This paper is organized as follows. Section 2 is devoted to the proof of a central result for this paper, a theorem defining the asymptotics of multiple integrals of a special type. In section 3, a brief review of the formulae for the deuteron form factors in the nonrelativistic and relativistic invariant impulse approximations is given. The deuteron form factors asymptotics is calculated in the nonrelativistic and relativistic impulse approximations with the help of the proven theorem in section 4. In section 5, asymptotics of the form factors is obtained for the deuteron wavefunctions in the conventional representation as a discrete superposition of Yukawa-type terms [9]. Section 6 contains the conclusions of this paper.

2. Theorem on the asymptotic expansion of some multiple integrals in the case when the maximal value of the integrand belongs to a region's boundary

In the following we will consider integrals of the kind:

$$F(\lambda) = \int_{\Omega} f(\lambda, x)\, \mathrm{e}^{S(\lambda,x)}\, \mathrm{d}x, \tag{1}$$

where Ω is a domain in $\mathbf{R}^n, x = (x_1, \ldots, x_n), \lambda$ is a large positive parameter. We will use the following definitions: $\partial\Omega$ is a boundary of the domain $\Omega, [\Omega] = \Omega \cup \partial\Omega$, the boundary $\partial\Omega \in C^\infty$ if in the vicinity of any point $x^0 \in \partial\Omega$ it can be specified by equation $x_j = \varphi(x'), x' \in U', x' = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n), U'$ is a neighborhood of a point $x^{'0}$, and the function $\varphi(x') \in C^\infty$ in U'.

We will consider function $S(\lambda, x)$, which has the maximal value at the point x^0. Note that we consider the function S with more general dependence on a large positive parameter λ as compared to classical books (see, e.g., [10]), where authors consider a product $\lambda S(x)$ usually. For functions under consideration $S(\lambda, x)$ there are asymptotic estimations in some particular cases only [11]. We stress that the difference $S(\lambda, x^0) - S(\lambda, x)$ increases with increasing parameter λ. This means that at $\lambda \to \infty$ the vicinity of the point x^0 makes the main contribution to the asymptotics of integrals (1). This qualitative statement we formulate as a lemma.

Lemma. *Let $S(\lambda, x)$ be a smooth function in $[\Omega]$, $f(\lambda, x)$ be a continuous function in $[\Omega]$, and $M(\lambda) \in C^1$,*

$$M(\lambda) = \sup_{x\in[\Omega]} S(\lambda, x), \tag{2}$$

at some $\lambda_0 > 0$ the integral (1) be absolutely convergent:

$$\int_\Omega |f(\lambda_0, x)|\, \mathrm{e}^{S(\lambda_0,x)}\, \mathrm{d}x < \infty, \tag{3}$$

and the following conditions be fulfilled at $\lambda \geqslant \lambda_0$:

$$\frac{\partial S(\lambda, x)}{\partial\lambda} \leqslant \frac{\mathrm{d}M(\lambda)}{\mathrm{d}\lambda}, \tag{4}$$

$$|f(\lambda, x)| \leqslant C_1 |f(\lambda_0, x)|. \tag{5}$$

Then at $\lambda \geqslant \lambda_0$ the following estimation is valid:

$$|F(\lambda)| \leqslant C_2\, \mathrm{e}^{M(\lambda)}. \tag{6}$$

Proof. At $\lambda \geqslant \lambda_0$ the following estimations are true:

$$\begin{aligned} |F(\lambda)| &\leqslant \mathrm{e}^{M(\lambda)} \int_\Omega \mathrm{e}^{S(\lambda_0,x)-M(\lambda_0)}\, \mathrm{e}^{S(\lambda,x)-S(\lambda_0,x)-M(\lambda)+M(\lambda_0)} |f(\lambda, x)|\, \mathrm{d}x \\ &\leqslant \mathrm{e}^{M(\lambda)-M(\lambda_0)} \int_\Omega \mathrm{e}^{S(\lambda_0,x)+S(\lambda,x)-S(\lambda_0,x)-M(\lambda)+M(\lambda_0)} |f(\lambda, x)|\, \mathrm{d}x. \end{aligned} \tag{7}$$

From conditions (4), (5) we obtain the inequality

$$|F(\lambda)| \leqslant C_1\, \mathrm{e}^{M(\lambda)-M(\lambda_0)} \int_\Omega \mathrm{e}^{S(\lambda_0,x)} |f(\lambda_0, x)|\, \mathrm{d}x \leqslant C_2\, \mathrm{e}^{M(\lambda)}. \tag{8}$$

Thus the statement (6) of the lemma is proven. □

Later we will consider function $S(\lambda, x)$ described in the lemma which has the maximal value at the point $x^0 \in \partial\Omega$, and $S(\lambda, x)$, $\partial\Omega \in C^\infty$ in the vicinity of x^0. This point is not the point of extremum, which means the validity of the following conditions,

$$\frac{\partial S(\lambda, x^0)}{\partial n} \neq 0, \tag{9}$$

and the matrix of coefficients B,

$$\left\| \frac{\partial^2 S(\lambda, x^0)}{\partial\xi_i \partial\xi_j} \right\|_{i,j=1}^{n-1} = B(\lambda), \tag{10}$$

gives the determined negative quadratic form. In equations (9), (10) $\partial/\partial n$ specifies the internal normal derivative $\vec{n}$ to the $\partial\Omega$, and $\xi_1, \ldots, \xi_{n-1}$ is an orthonormal basis in the tangential to the $\partial\Omega$ plane $T\partial\Omega_{x^0}$ at the x^0 point.

For convenience let us choose in the vicinity of point x^0 a frame $y = (y_1, \ldots, y_n)$, so that x^0 is the origin of coordinate and the internal normal to $\partial\Omega$ coincides with the last basis vector of the new coordinate system. We denote functions f, S in these coordinates as f^*, S^*, and U^* is an image of U (that is an image of a half-vicinity of the point x^0). The equation for ∂U^* in the vicinity of the point $y = 0$ can be written in the following way:

$$y_n = \varphi(y'), \qquad y' \in U', \qquad y' = (y_1, \ldots, y_{n-1}), \tag{11}$$

with U' being a vicinity of the point $y' = 0$, $\varphi(y') \in C^\infty(U')$, and at $y' \to 0$, $\varphi(y') = O(|y'|^2)$.

Theorem. *Let the following conditions be fulfilled:*

1°. $f, S \in C([\Omega])$.
2°. $S(\lambda, x^0)$ *is a maximal value of function* $S(\lambda, x)$, $x^0 \in \partial\Omega$, *and* x^0 *is not a point of extremum.*
3°. $f, S, \partial\Omega \in C^\infty$ *in the vicinity of the point* x^0.
4°. The Taylor expansion of functions S^* *and* f^* *in the vicinity of point* x^0 *satisfies the following relations:*

$$f^*(\lambda, y', \varphi(y')) = f^*(\lambda, 0, \varphi(0))[1 + o(1)], \tag{12}$$

$$S^*(\lambda, y', \varphi(y')) - S^*(\lambda, 0, \varphi(0)) = \tfrac{1}{2}\langle B(\lambda)y', y'\rangle + O(|y'|^3), \tag{13}$$

the angle brackets denote bilinear form: $\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$.
Then at $\lambda \to \infty$ *the following asymptotic expansion is valid:*

$$F(\lambda) \sim \exp[S(\lambda, x^0)] \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} h_{km}(\lambda). \tag{14}$$

The way of calculating coefficients $h_{km}(\lambda)$ *will be determined later.*

Proof. Let us divide the integral (1) into two integrals. The integration domain of the first one is the half-vicinity U of the point x^0, and the integration domain of the second one is a remainder of integration domain of the original integral. As we stressed above $M(\lambda)$ (2) for the second integral is significantly smaller as compared to the $M(\lambda)$ for the first one at $\lambda \to \infty$. And it follows immediately from the proven lemma that the second integral is exponentially small as compared to the first one which is proportional to $\exp[S(\lambda, x^0)]$. So we will estimate asymptotically the first integral only.

In the expansion of the function $S^*(\lambda, y', \varphi(y'))$ in line with the condition (13) linear components are absent, because the point $y' = 0$ is a point of maximum of the function $S^*(\lambda, y', \varphi(y'))$ in the region U'.

Let us choose U in accordance with inequalities $\varphi(y') \leqslant y_n \leqslant \delta$, $\delta > 0$ at $y \in U^*$. Then we can represent the integral (1) within exponentially decreasing terms:

$$F(\lambda) = \int_{U^*} f^*(\lambda, y', y_n) \exp[S^*(\lambda, y', y_n)]\, dy. \tag{15}$$

Let us rewrite integral (15) in the following way:

$$F(\lambda) = \int_{U'} \Phi(\lambda, y')\, dy', \tag{16}$$

with

$$\Phi(\lambda, y') = \int_{\varphi(y')}^{\delta} \exp[S^*(\lambda, y', y_n)] f^*(\lambda, y', y_n)\, dy_n. \tag{17}$$

The integral (17) is one-dimensional, and the function $S^*(\lambda, y', y_n)$ reaches maximal value on the boundary $y_n = \varphi(y')$. An asymptotic expansion of this integral can be found through integration by parts. After $N + 1$ integrations we obtain the sequence

$$\Phi(\lambda, y') = \sum_{k=0}^{N} M^k \left[\frac{f^*(\lambda, y', y_n)}{S^{*'}(\lambda, y', y_n)}\right] \exp[S^*(\lambda, y', y_n)]|_{\varphi(y')}^{\delta} + \int_{\varphi(y')}^{\delta} S^{*'}(\lambda, y', y_n) M^{N+1} \left[\frac{f^*(\lambda, y', y_n)}{S^{*'}(\lambda, y', y_n)}\right] \exp[S^*(\lambda, y', y_n)]\, \mathrm{d}y_n, \tag{18}$$

with M^0 being a unit operator and

$$M^k = \left(-\frac{1}{S^{*'}(\lambda, y', y_n)} \frac{\mathrm{d}}{\mathrm{d}y_n}\right)^k. \tag{19}$$

The substitution of $y_n = \varphi(y')$ provides the main contribution to the asymptotics, the value of $y_n = \delta$ is exponentially small at $\lambda \to \infty$ as compared with the previous. Further integration under these conditions gives the following expansion for the function (17):

$$\Phi(\lambda, y') \sim -\exp[S^*(\lambda, y', \varphi(y'))] \sum_{k=0}^{\infty} M^k \left[\frac{f^*(\lambda, y', \varphi(y'))}{S^{*'}(\lambda, y', \varphi(y'))}\right]. \tag{20}$$

So

$$F(\lambda) \sim -\sum_{k=0}^{\infty} \int_{U'} \exp[S^*(\lambda, y', \varphi(y'))] M^k \left[\frac{f^*(\lambda, y', \varphi(y'))}{S^{*'}(\lambda, y', \varphi(y'))}\right] \mathrm{d}y'. \tag{21}$$

The point $y' = 0$ is an internal point of maximum of the integrand in the expression (21). Functions $S^*(\lambda, y', \varphi(y'))$ and $f^*(\lambda, y', \varphi(y'))$ satisfy the conditions of lemma (4) and theorem (12), (13), therefore we can apply some estimations. One can rewrite (21) as,

$$F(\lambda) \exp[-S^*(\lambda, 0, \varphi(0))] \sim -\sum_{k=0}^{\infty} \int_{U'} H_k(\lambda, y') \exp\left[-\frac{1}{2}\langle -B(\lambda) y', y'\rangle\right] \mathrm{d}y', \tag{22}$$

where

$$H_k(\lambda, y') = M^k \left[\frac{f^*(\lambda, y', \varphi(y'))}{S^{*'}(\lambda, y', \varphi(y'))}\right] \times \exp\left[S^*(\lambda, y', \varphi(y')) - S^*(\lambda, 0, \varphi(0)) - \frac{1}{2}\langle B(\lambda) y', y'\rangle\right]. \tag{23}$$

Let us transform identically the terms of series (22) using the Parseval equality $\int g(x)h(x)\, \mathrm{d}x = (2\pi)^{-n} \int \tilde{g}(\xi)\tilde{h}(\xi)\, \mathrm{d}\xi$, where $\tilde{g}(\xi), \tilde{h}(\xi)$ are Fourier transforms of $g(x), h(x)$ respectively,

$$\begin{aligned} F(\lambda) &\exp[-S^*(\lambda, 0, \varphi(0))] \\ &\sim -\sum_{k=0}^{\infty} (2\pi)^{-\frac{n-1}{2}} |\det B(\lambda)|^{-1/2} \int_{U'_\xi} \tilde{H}_k(\lambda, \xi) \exp\left[-\frac{1}{2}\langle -B(\lambda)^{-1}\xi, \xi\rangle\right] \mathrm{d}\xi \\ &= -\sum_{k=0}^{\infty} (2\pi)^{\frac{n-1}{2}} |\det B(\lambda)|^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!} \\ &\quad \times \int_{U'} H_k(\lambda, y') \langle -B(\lambda)^{-1} \delta^{(m)}(y'), \delta^{(m)}(y')\rangle^m \,\mathrm{d}y' \\ &= -\sum_{k=0}^{\infty} (2\pi)^{\frac{n-1}{2}} |\det B(\lambda)|^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!} \langle B(\lambda)^{-1} \nabla, \nabla\rangle^m H_k(\lambda, y')|_{y'=0} \end{aligned} \tag{24}$$

where $\tilde{H}_k(\lambda, \xi)$ is a Fourier transform of $H_k(\lambda, y')$, $\delta(y')$ is a Dirac delta function.

So we can write the asymptotic series for the integral (15):

$$F(\lambda) \sim e^{S^*(\lambda,0,0)} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} h_{km}(\lambda), \tag{25}$$

$$h_{km}(\lambda) = (2\pi)^{\frac{n-1}{2}} |\det B(\lambda)|^{-1/2} \frac{(-1)^{m+1}}{2^m m!} \langle B(\lambda)^{-1}\nabla, \nabla\rangle^m H_k(\lambda, y')|_{y'=0}. \tag{26}$$

Thus the theorem is proven. □

The first asymptotic term in the x variable

$$F(\lambda) \sim -(2\pi)^{\frac{n-1}{2}} \exp[S(\lambda, x^0)] \left(\frac{\partial S(\lambda, x^0)}{\partial n}\right)^{-1} |\det B(\lambda)|^{-\frac{1}{2}} f(\lambda, x^0), \tag{27}$$

where $\vec{n}$, $B(\lambda)$ are defined by conditions (9) and (10).

3. Electromagnetic deuteron form factors in the nonrelativistic and relativistic impulse approximations

In the nonrelativistic impulse approximation known formulae for electromagnetic deuteron form factors can be rewritten in the following way [12]:

$$\begin{aligned} G_C^{NR}(Q^2) &= \sum_{l,l'} \int k^2\, dk\, k'^2\, dk' u_l(k) \tilde{g}_{0C}^{ll'}(k, Q^2, k') u_{l'}(k'), \\ G_Q^{NR}(Q^2) &= \frac{2M_d^2}{Q^2} \sum_{l,l'} \int k^2\, dk\, k'^2\, dk' u_l(k) \tilde{g}_{0Q}^{ll'}(k, Q^2, k') u_{l'}(k'), \\ G_M^{NR}(Q^2) &= -M_d \sum_{l,l'} \int k^2\, dk\, k'^2\, dk' u_l(k) \tilde{g}_{0M}^{ll'}(k, Q^2, k') u_{l'}(k'). \end{aligned} \tag{28}$$

Here $u_l(k)$ are the deuteron wavefunctions in the momentum representation, $l, l' = 0, 2$ are orbital angular momenta, $\tilde{g}_{0i}^{ll'}(k, Q^2, k'), i = C, Q, M$ are nonrelativistic free two-particle charge, quadrupole and magnetic dipole form factors respectively, M_d is the deuteron mass. Formulae for $\tilde{g}_{0i}^{ll'}$ are given in [8].

Let us discuss briefly the possible types of the model, deuteron wavefunctions. There are several classes of the deuteron wavefunctions: obtained with microscopic model Hamiltonians of the NN–interaction in the nonrelativistic nuclear physics (for example, see [9]), deduced from scattering amplitudes in the Bethe–Salpeter approach and its various quasipotential reductions (see [13]), wavefunctions of the Poincaré-invariant quantum mechanics (as an example see wavefunctions in the instant form of PIQM [4–8]), and also wavefunctions calculated in the various statements of inverse scattering problems [14–16]. But independently of the method any wavefunction can be represented as the following Laguerre polynomial expansion [16]:

$$u_l(k) = \sum_{m=0}^{\infty} a_{lm} \sqrt{\frac{2m!}{\Gamma(m+l+3/2)}} r_0^{l+\frac{3}{2}} k^l L_m^{l+\frac{1}{2}} \left(r_0^2 k^2\right) e^{-\frac{r_0^2 k^2}{2}} \tag{29}$$

or in the coordinate representation,

$$u_l(r) = \sum_{m=0}^{\infty} (-1)^m a_{lm} \sqrt{\frac{2m!}{r_0 \Gamma(m+l+3/2)}} \left(\frac{r}{r_0}\right)^{l+1} L_m^{l+\frac{1}{2}} \left(\frac{r^2}{r_0^2}\right) e^{-\frac{r^2}{2r_0^2}}, \tag{30}$$

here $L_m^{l+1/2}(x)$ are generalized Laguerre polynomials, $\Gamma(x)$ is an Euler gamma function, the dimensional parameter r_0 can be related to the deuteron matter radius (see section 5).

The wavefunction representation as a Laguerre polynomial expansion (29) is very useful for the calculation of the asymptotic behavior of the form factors. However, one can avoid such a representation and obtain the asymptotic expansion directly for the initial wavefunction.

Generally, at a high transferred momentum it is necessary to take into account relativistic corrections in the electromagnetic deuteron structure. In our paper the relativistic description of the deuteron is constructed in the framework of the instant form of Poincaré-invariant quantum mechanics (PIQM), developed by authors previously [4–8]. In this approach we present the electromagnetic deuteron form factors by analogy with the nonrelativistic case (28). The corresponding formulae in the relativistic impulse approximation were obtained in our paper [6]:

$$
\begin{aligned}
G_C^R(Q^2) &= \sum_{l,l'} \int d\sqrt{s}\, d\sqrt{s'} \varphi_l(s) g_{0C}^{ll'}(s, Q^2, s')\varphi_{l'}(s'),\\
G_Q^R(Q^2) &= \frac{2M_d^2}{Q^2} \sum_{l,l'} \int d\sqrt{s}\, d\sqrt{s'} \varphi_l(s) g_{0Q}^{ll'}(s, Q^2, s')\varphi_{l'}(s'),\\
G_M^R(Q^2) &= -M_d \sum_{l,l'} \int d\sqrt{s}\, d\sqrt{s'} \varphi_l(s) g_{0M}^{ll'}(s, Q^2, s')\varphi_{l'}(s'),
\end{aligned}
\tag{31}
$$

where $\varphi_l(s)$ are the deuteron wavefunctions in the sense of PIQM, $g_{0i}^{ll'}((s, Q^2, s'), i = C, Q, M$ are relativistic free two-particle charge, quadrupole and magnetic dipole form factors respectively. Formulae for free form factors are given in [8].

So the deuteron wavefunctions in the sense of PIQM are solutions of eigenvalue problem for a mass squared operator for the deuteron (see, e.g. [4]): $\hat{M}_d^2|\psi\rangle = M_d^2|\psi\rangle$. An eigenvalue problem for this operator is coincident with the nonrelativistic Schrödinger equation within a second order on deuteron binding energy $\varepsilon_d^2\big/(4M)$, the value of which is small (M is an averaged nucleon mass). So the deuteron wavefunctions in the sense of PIQM differ from the nonrelativistic wavefunctions by conditions of normalization only. In the relativistic case the wavefunctions are normalized with relativistic density of states:

$$
\begin{aligned}
&\sum_{l=0,2} \int_0^\infty \varphi_l^2(k) \frac{\mathrm{d}k}{2\sqrt{k^2+M^2}} = 1,\\
&\varphi_l(k) = \sqrt[4]{s}\, k u_l(k), \qquad s = 4(k^2+M^2).
\end{aligned}
\tag{32}
$$

The nonrelativistic formulae (28) can be obtained from relativistic ones (31) in the nonrelativistic limit. This limiting procedure can be performed in the most natural way in the instant form of PIQM. The reason is that in papers [4–8] we have constructed the successful formalism of the instant form of PIQM. In the case of other forms of PIQM (point and front forms) obtaining nonrelativistic limits is much more difficult.

For obtaining the asymptotic form factors' behavior at high transferred momentum in the nonrelativistic and relativistic cases it is necessary to estimate asymptotically double integrals (28) and (31) at $Q^2 \to \infty$. Note that the integrands reach their maximum values at the integration domain boundary, and these points are not points of extremum. In the previous section the theorem defining asymptotics of n-tuple integrals of such kinds was proven.

4. Asymptotic expansion of the deuteron form factors

We start the asymptotic expansion of the deuteron form factors from the nonrelativistic case. It is caused by the simplicity of the nonrelativistic formulae, so the calculation of the asymptotics is more clear. In what follows the relativistic calculation will be presented analogously to the nonrelativistic one, although more cumbersomely. Moreover, the nonrelativistic calculation is interesting because the nonrelativistic formulae for the form factors (28) are conventional, which is why their correct asymptotic expansion has universal appeal. Let us emphasize also that the relativistic expressions for form factors and, therefore, their asymptotic estimations depend on the choice of the method of relativisation of the two-nucleon model. Nonrelativistic calculation is also of interest because it helps to clarify the role of relativistic effects in the electromagnetic structure of the deuteron at the asymptotical momentum transfers.

As we have seen in section 3, the deuteron form factors in the nonrelativistic impulse approximation can be represented by double integrals (28). We will find its asymptotic expansion using the theorem of section 2 and use as an example the asymptotics of the charge form factor. We shall estimate only the $l = l' = 0$ term in the sum (28) because the asymptotics of the other terms of form factors (28) can be derived analogously.

Let us rewrite the corresponding $l = l' = 0$ term of the charge form factor (28) using (29):

$$\int \left(\sum_m a_{0m}\sqrt{\frac{2m!}{\Gamma(m+3/2)}} r_0^{\frac{3}{2}} L_m^{\frac{1}{2}}\left(r_0^2k^2\right)\right)\left(\sum_m a_{0m}\sqrt{\frac{2m!}{\Gamma(m+3/2)}} r_0^{\frac{3}{2}} L_m^{\frac{1}{2}}\left(r_0^2k'^2\right)\right) \times \tilde{g}_{0C}^{00}(k, Q^2, k') \exp[S(k,k')]k^2\,\mathrm{d}k\,k'^2\,\mathrm{d}k'. \tag{33}$$

We have denoted in (33):

$$S(k,k') = -\frac{r_0^2}{2}(k^2 + k'^2). \tag{34}$$

The expression for $\tilde{g}_{0C}^{00}(k, Q^2, k')$ is commonly accepted (see, e.g., [8]):

$$\tilde{g}_{0C}^{00}(k, Q^2, k') = \frac{1}{kk'Q}\left(G_E^p(Q^2) + G_E^n(Q^2)\right) \times \left[\theta\left(k' - \left|k - \frac{Q}{2}\right|\right) - \theta\left(k' - k - \frac{Q}{2}\right)\right], \tag{35}$$

$G_E^{p,n}(Q^2)$ are electric form factors of proton and neuteron respectively, $\theta(x)$ is a step function.

In the case under consideration the space dimension $n = 2$, $(x_1, x_2) = (k, k')$, $\lambda = Q^2$ is a large positive parameter. The integration domain is determined by θ-functions in (35) and is shown in figure 1. The location of the point of maximal value of the function S can be obtained by analysis of (34) and (35): $(k^0, k'^0) = \left(\frac{Q}{4}, \frac{Q}{4}\right)$.

Let us perform the transition to the new basis as we have described before. We perform the shift of the origin of coordinates to the point of maximal value of the function S. Then we rotate the obtained frame for the internal normal to the boundary in the new origin to be coincident with the last basis vector of the new frame. This procedure is illustrated in figure 1.

In other words we perform the transition to the new variables in (33):

$$k = \frac{Q}{\sqrt{2}}(t' + t) + \frac{Q}{4}, \qquad k' = \frac{Q}{\sqrt{2}}(t' - t) + \frac{Q}{4}. \tag{36}$$

At this transformation function $S(k, k')$ gets dependent on large parameter Q^2:

$$S^*(Q^2, t, t') = -\frac{Q^2 r_0^2}{2}\left(t^2 + t'^2 + \frac{t'}{\sqrt{2}} + \frac{1}{8}\right). \tag{37}$$

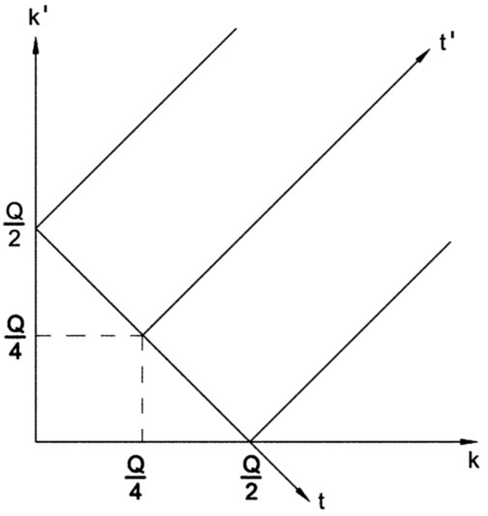

Figure 1. The integration domain, location of the point of maximal value and transition to the new variables for the nonrelativistic case.

Functions $S^*(Q^2, t, t')$, $\tilde{g}_{0C}^{00}(k, Q^2, k')$, and the boundary of the integration domain satisfy the conditions 1°, 3°, 4° of the theorem. The location of the point, which satisfies the conditions (9), (10), can be obtained by a simple analysis of the function (34): $(t^0, t'^0) = (0, 0)$. Let us show that this point satisfies condition 2° of the theorem.

It is obvious, that at the point of maximal value

$$\frac{\partial S}{\partial n} = \frac{\partial S^*}{\partial t'}\bigg|_{(t,t')=(0,0)} = -\frac{r_0^2 Q^2}{2\sqrt{2}} \neq 0. \tag{38}$$

So condition (9) is satisfied.

Let us calculate now the $B(\lambda)$ matrix from condition (10). In our case the tangent to the domain of the integration boundary vector in the point of maximal value is $\vec{\xi} = (1/\sqrt{2}, -1/\sqrt{2})$, i.e. the $B(\lambda)$ matrix is a number:

$$\frac{\partial^2 S}{\partial \xi^2} = \frac{\partial^2 S^*}{\partial t^2}\bigg|_{(t,t')=(0,0)} = -r_0^2 Q^2 < 0. \tag{39}$$

We note that $B(\lambda)$ is negative-definite, i.e. the point $(t^0, t^{0\prime}) = (0, 0)$ is really the point of maximal value. So the point (0,0) satisfies condition 2° of the theorem.

So integral (33) satisfies the requirements of the theorem proven in section 2. Therefore we can apply the asymptotic formula (14).

Calculating by analogy the other terms of the sum (28) we obtain asymptotic expansions of deuteron form factors in the nonrelativistic impulse approximation:

$$G_i^{NR}(Q^2) \sim A_i\, \mathrm{e}^{-\frac{r_0^2 Q^2}{16}} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} h_{km}^{NR}, \tag{40}$$

$$h_{km}^{NR} = \frac{\sqrt{2\pi}}{2^m m!} \frac{1}{(Qr_0)^{2k+2m+3}} \sum_{p=0}^{k} b_{kp} (2\sqrt{2})^{k+p+1} \frac{\partial^{2m}}{\partial t^{2m}} f_i^{NR(k-p)}(t, Q^2, 0)\bigg|_{t=0}, \tag{41}$$

$$b_{k0} = 1, \quad b_{k+1p} = b_{kp} - (k+p) b_{kp-1}, \quad (p < k), \quad b_{kk} = (-1)^k (2k-1)!!, \tag{42}$$

$$f_i^{NR}(t,Q^2,t') = \sum_{l,l'=0,2} Q^2k^2k'^2\tilde{u}_l(k)\tilde{g}_{0i}^{ll'}(t,Q^2,t')\tilde{u}_{l'}(k'), \tag{43}$$

with $k = k(t,Q^2,t')$, $k' = k'(t,Q^2,t')$, variables t,t' are denoted in (36), $i = C,Q,M$, $A_C = 1$, $A_Q = 2M_d^2/Q^2$, $A_M = -M_d$,

$$f_i^{NR(m)}(t,Q^2,t') = \frac{\partial^m}{\partial t'^m} f_i^{NR}(t,Q^2,t'). \tag{44}$$

$\tilde{u}_{l,l'}$ are defined by equalities (see (29), too):

$$u_0(k) = \tilde{u}_0(k)\,e^{-\frac{r_0^2k^2}{2}}, \qquad u_2(k) = \tilde{u}_2(k)\,e^{-\frac{r_0^2k^2}{2}}. \tag{45}$$

Let us now perform the calculation of the relativistic asymptotics of deuteron form factors. To estimate asymptotically integrals (31) we proceed analogously to the nonrelativistic case, i.e. we use relativistic analogs of the corresponding nonrelativistic formulae (33)–(39). Now the free relativistic charge form factor in (31) at $l = l' = 0$ is given in [8]:

$$\begin{aligned} g_{0C}^{00}(s,Q^2,s') = R(s,Q^2,s')Q^2\Big[&(s+s'+Q^2)\left(G_E^p(Q^2)+G_E^n(Q^2)\right)g_{CE}^{00} \\ &+\frac{1}{M}\xi(s,Q^2,s')\left(G_M^p(Q^2)+G_M^n(Q^2)\right)g_{CM}^{00}\Big], \end{aligned} \tag{46}$$

$G_{E,M}^{p,n}(Q^2)$ are electric and magnetic form factors of proton and neutron respectively,

$$\begin{aligned} g_{CE}^{00} &= \left(\frac{1}{2}\cos\omega_1\cos\omega_2 + \frac{1}{6}\sin\omega_1\sin\omega_2\right), \\ g_{CM}^{00} &= \left(\frac{1}{2}\cos\omega_1\sin\omega_2 - \frac{1}{6}\sin\omega_1\cos\omega_2\right), \end{aligned} \tag{47}$$

$$R(s,Q^2,s') = \frac{(s+s'+Q^2)}{\sqrt{(s-4M^2)(s'-4M^2)}}\,\frac{\vartheta(s,Q^2,s')}{[\lambda(s,-Q^2,s')]^{3/2}}\,\frac{1}{\sqrt{1+Q^2/4M^2}}, \tag{48}$$

$$\xi(s,Q^2,s') = \sqrt{ss'Q^2 - M^2\lambda(s,-Q^2,s')}, \tag{49}$$

ω_1 and ω_2 are angles of the Wigner spin rotation,

$$\begin{aligned} \omega_1 &= \arctan\frac{\xi(s,Q^2,s')}{M[(\sqrt{s}+\sqrt{s'})^2+Q^2]+\sqrt{ss'}(\sqrt{s}+\sqrt{s'})}, \\ \omega_2 &= \arctan\frac{\alpha(s,s')\xi(s,Q^2,s')}{M(s+s'+Q^2)\alpha(s,s')+\sqrt{ss'}(4M^2+Q^2)}, \end{aligned} \tag{50}$$

where $\alpha(s,s') = 2M + \sqrt{s} + \sqrt{s'}$, $\vartheta(s,Q^2,s') = \theta(s'-s_1) - \theta(s'-s_2)$, θ is a step function, $\lambda(a,b,c) = a^2+b^2+c^2-2(ab+ac+bc)$,

$$s_{1,2} = 2M^2 + \frac{1}{2M^2}(2M^2+Q^2)(s-2M^2) \mp \frac{1}{2M^2}\sqrt{Q^2(Q^2+4M^2)s(s-4M^2)}. \tag{51}$$

To obtain a relativistic asymptotic expansion we also perform transition to the new basis (shift and rotation analogously to (36)). The function S and the boundary of the integration domain differ from nonrelativistic ones, so it is necessary to perform a special analysis. In other words, instead of a change of variables (36) we perform the following replacement:

$$\begin{aligned} s &= \frac{Q^2}{\sqrt{2}}(t'+t) + 2M^2 + M\sqrt{Q^2+4M^2}, \\ s' &= \frac{Q^2}{\sqrt{2}}(t'-t) + 2M^2 + M\sqrt{Q^2+4M^2}. \end{aligned} \tag{52}$$

At this transformation one could obtain the function $S^*(Q^2,t,t')$:

$$S^*(Q^2,t,t') = -\frac{Q^2 r_0^2}{2}\left(\frac{t'}{2\sqrt{2}} + \frac{M}{2Q}\sqrt{1+\frac{4M^2}{Q^2}} - \frac{M^2}{Q^2}\right). \tag{53}$$

The point of maximal value is $(t^0, t'^0) = (0, 0)$, and subject to boundary equation $t' = t'(t)$ at this point

$$\frac{\partial S}{\partial n} = \frac{\partial S^*}{\partial t'}\bigg|_{(t,t')=(0,0)} = -\frac{r_0^2 Q^2}{4\sqrt{2}}, \qquad \frac{\partial^2 S}{\partial \xi^2} = \frac{\partial^2 S^*}{\partial t^2}\bigg|_{(t,t')=(0,0)} = -\frac{r_0^2 Q^3}{8M}. \tag{54}$$

Functions $S^*(Q^2,t,t')$, $g_{0C}^{00}(s,Q^2,s')$, the point $(t^0,t'^0) = (0,0)$ and the boundary of the integration domain satisfy the conditions 1°–4° of the theorem. Therefore we can apply the asymptotic formula (14).

So we obtain asymptotic expansion of the relativistic deuteron form factors:

$$G_i^R(Q^2) \sim A_i\,\mathrm{e}^{-\frac{r_0^2}{2}\left(\frac{M}{2}\sqrt{Q^2+4M^2}-M^2\right)} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} h_{km}^R \tag{55}$$

$$h_{km}^R = \sqrt{\pi} \sum_{p=0}^{m/2+[(-1)^m-1]/4} \frac{1}{Q^{2k+3m-5p+\frac{7}{2}}} \frac{2^{2m+\frac{5}{2}k-7p+\frac{9}{2}}}{r_0^{2m+2k-2p+3} M^{3p-m-\frac{1}{2}}} C_{4p}^{2m} \frac{(4p)!}{p!m!}$$

$$\times \frac{\partial^{2m-4p}}{\partial t^{2m-4p}} f_i^{R(k)}(t,Q^2,\varphi(t))\bigg|_{t=0}, \tag{56}$$

$$f_i^R(t,Q^2,t') = \sum_{l,l'=0,2} \tilde{u}_l(k) g_{0i}^{ll'}(t,Q^2,t')\tilde{u}_{l'}(k') Q^4 \frac{(s/4-M^2)^{\frac{1}{2}}(s'/4-M^2)^{\frac{1}{2}}}{4\sqrt[4]{ss'}}, \tag{57}$$

$$f_i^{R(m)}(t,Q^2,t') = \frac{\partial^m}{\partial t'^m} f_i^R(t,Q^2,t'). \tag{58}$$

Functions $k = k(s), k' = k'(s')$ are specified in (32), $s = s(t,Q^2,t'), s' = s'(t,Q^2,t')$, variables t, t' are denoted in (52), C_{4p}^{2m} are the binomial coefficients.

Expansions (40) and (55) are the asymptotic series in reciprocal powers of the parameter Q with known coefficients. The asymptotic expansion of this type for the deuteron form factors is obtained in this work for the first time.

One can see from formulae (40) and (55) that relativistic corrections change the behavior of form factors at high momentum transfer. In particular, the exponential multiplier index is Q^2 in the nonrelativistic case, but in the relativistic case it is Q at $Q^2 \to \infty$. It seems to be a general feature of our relativistic approach to the description of composite systems; in particular, we have obtained the similar result in consideration of asymptotic behavior of the pion form factor in the composite quark model [17].

5. Asymptotics of the form factors for the conventional wavefunctions representation

In this section we represent the obtained asymptotic expansions (40) and (55) in terms of initial wavefunctions on the left-hand side of (29), (32). For this representation it is necessary to replace functions $\tilde{u}_l(k)$ by functions $u_l(k)$ in (40) and (55) using (32), (45). Keeping the leading term on $1/Q$ in the asymptotic expansions (40) and (55) one can obtain the following asymptotic formulae in terms of functions $u_l(k)$ and $\varphi_l(s)$ from (29), (32):

$$G_i^{NR}(Q^2) \sim A_i \frac{4\sqrt{\pi}}{r_0^3 Q} \sum_{l,l'=0,2} k^2 k'^2 u_l(k) \tilde{g}_{0i}^{ll'}(t, Q^2, t') u_{l'}(k') \Big|_{\substack{t=0\\t'=0}} \tag{59}$$

$$G_i^{R}(Q^2) \sim A_i \frac{16\sqrt{2\pi M Q}}{r_0^3} \frac{(s/4 - M^2)^{\frac{1}{2}}(s'/4 - M^2)^{\frac{1}{2}}}{4\sqrt[4]{ss'}} \sum_{l,l'=0,2} u_l(k) g_{0i}^{ll'}(t, Q^2, t') u_{l'}(k') \Big|_{\substack{t=0\\t'=0}}. \tag{60}$$

Let us note that a similar asymptotic representation can be obtained for any finite number of terms in the asymptotic expansions (40), (55).

In the modern calculations the deuteron wavefunctions are usually represented as a discrete superposition of Yukawa-type terms (see, e.g., [9]):

$$u_0(k) = \sqrt{\frac{2}{\pi}} \sum_j \frac{C_j}{\left(k^2 + m_j^2\right)}, \qquad u_2(k) = \sqrt{\frac{2}{\pi}} \sum_j \frac{D_j}{\left(k^2 + m_j^2\right)}, \tag{61}$$

or in the coordinate representation,

$$\begin{aligned} u_0(r) &= \sum_j C_j \exp(-m_j r), \\ u_2(r) &= \sum_j D_j \exp(-m_j r) \left[1 + \frac{3}{m_j r} + \frac{3}{(m_j r)^2}\right], \\ m_j &= \alpha + m_0(j-1), \qquad \alpha = \sqrt{M|\varepsilon_d|}. \end{aligned} \tag{62}$$

Coefficients C_j, D_j, maximal value of the index j and m_0 are determined by the best fit of the corresponding solution of Schrödinger equation.

The deuteron wavefunctions analytical form (62) results in the right behavior of the wavefunctions at large distances:

$$u_0(r) \sim \exp(-\alpha r), \qquad u_2(r) \sim \exp(-\alpha r)\left(1 + \frac{3}{(\alpha r)} + \frac{3}{(\alpha r)^2}\right). \tag{63}$$

The deuteron wavefunctions behavior at small distances:

$$u_0(r) \sim r, \qquad u_2(r) \sim r^3, \tag{64}$$

is provided by imposing the following conditions on coefficients C_j and D_j:

$$\sum_j C_j = 0, \qquad \sum_j D_j = \sum_j D_j m_j^2 = \sum_j \frac{D_j}{m_j^2} = 0. \tag{65}$$

Let us substitute the wavefunctions (61) into (59) and (60), and then obtain the leading asymptotic terms of the nonrelativistic deuteron form factors:

$$G_C^{NR} \sim \frac{1}{Q^8} \frac{2^{15}}{\sqrt{\pi} r_0^3} \left[\sum_j C_j m_j^2\right]^2 \left(G_E^p(Q^2) + G_E^n(Q^2)\right), \tag{66}$$

$$G_Q^{NR} \sim 3M_d^2 \frac{1}{Q^{12}} \frac{2^{\frac{41}{2}}}{\sqrt{\pi} r_0^3} \left[\sum_j C_j m_j^2\right] \left[\sum_j D_j m_j^4\right] \left(G_E^p(Q^2) + G_E^n(Q^2)\right), \tag{67}$$

$$G_M^{NR} \sim \frac{1}{Q^8} \frac{2^{15} M_d}{\sqrt{\pi} r_0^3 M} \left[\sum_j C_j m_j^2\right]^2 \left(G_M^p(Q^2) + G_M^n(Q^2)\right). \tag{68}$$

The dimensional parameter r_0 can be found from the expression for the deuteron matter radius in our deuteron model:

$$r_m^2 = \frac{1}{4}\int_0^\infty \left(u_0^2(r) + u_2^2(r)\right)r^2\,\mathrm{d}r. \tag{69}$$

One can substitute wavefunctions of the form (30) into this expression. So formula (69) specifies an algebraic equation for r_0. The solution of this equation can be found numerically.

It should be pointed out that the main terms of the expansion of charge and magnetic form factors in (66)–(68) are determined by the S-state of deuteron only. The D-wave function gives the contribution to the main term of the quadrupole from factor. Its faster decrease at $Q^2 \to \infty$ in comparison to the other form factors is a consequence of the faster decrease of the D-wave function at small distances in comparison to the S-wave (64). From the mathematical point of view the type of leading terms in (66)–(68) is a consequence of conditions on the coefficients (65). The modification of these conditions obviously results in change of the main terms in (66)–(68). From these formulae it is also noted that the asymptotic expansions for the deuteron form factors contain dependence on the asymptotics of nucleon form factors.

We emphasize that in the other deuteron asymptotics investigations only the power dependence on the transferred momentum was calculated as a rule. In the present paper we give a rigorous calculation of a multiplicative preasymptotic constant.

One can calculate the relativistic asymptotics of the form factors by analogy with the nonrelativistic case. For this calculation we use formulae (32), (60), (65). As a result we obtain

$$G^R_{C,M}(Q^2) \sim \frac{Q^3}{2^{\frac{11}{2}}M^3}G^{NR}_{C,M}(Q^2), \tag{70}$$

$$G^R_Q(Q^2) \sim \frac{Q^4}{2^{\frac{15}{2}}M^4}G^{NR}_Q(Q^2). \tag{71}$$

Note that the asymptotic expansions (66)–(68) and (70), (71) are obtained for the first time in our work. It is interesting to compare the obtained asymptotic estimations (66)–(68), (70), (71) with observable behavior of the deuteron characteristics. In the experiment at high momentum transfer one measures a combination of electromagnetic form factors, for example, the structure function $A(Q^2) = G_C^2(Q^2) + \frac{8}{9}\eta^2 G_Q^2(Q^2) + \frac{2}{3}\eta G_M^2(Q^2)$, where $\eta = Q^2/4M_d^2$. This function enters the differential cross section of the elastic ed scattering. The values of function $A(Q^2)$ are known up to $Q^2 \simeq 6$ (GeV c^{-1})^2. For comparison with the experimental data one needs to specify asymptotics of the nucleon form factors. It is natural to choose for nucleon form factors the asymptotic which is predicted by the quark model [1] $G_M^{p,n} \sim 1/Q^4$. Under these conditions the power dependence on Q^2 of the function $A(Q^2)$ in the relativistic impulse approximation coincides with the experimentally observed one. The physical consequences will be examined in detail in another paper.

6. Conclusion

A theorem defining asymptotics of multiple integrals of some special type is proven. With the help of the theorem the asymptotic expansion of the deuteron electromagnetic form factors at $Q^2 \to \infty$ is calculated for the first time. The expansion is represented as an asymptotic series in inverse powers of momentum transfer. The asymptotics of the form factors is found in terms of the conventional representation of the deuteron wave function as a discrete superposition of Yukawa-type terms. The asymptotic behavior of the form factors is calculated in the nonrelativistic impulse approximation and in the relativistic invariant impulse approximation proposed by the authors in the instant form of the Poincaré-invariant quantum mechanics previously. It is established that relativistic corrections change the power dependence of the form factors on the momentum transfer at $Q^2 \to \infty$ and slow down its decrease. It is also found that relativistic effects result in the agreement of the theoretical asymptotics and the experimentally observed behavior of the structure function $A(Q^2)$ at highest achieved momentum transfers.

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